Pergamon

PII: S0021-8928(96)00073-1

J. Appl. Maths Mechs, Vol. 60, No. 4, pp. 579–590, 1996

0021-8928/96 \$24.00+0.00

THE STABILITY OF STOCHASTICALLY PERTURBED ORBITAL MOTIONS†

L. B. RYASHKO

Ekaterinburg

(*Received* 26 *January* 1995)

When investigating the orbital stability of non-linear stochastic systems, two forms of first-approximation systems (with noise of types I and II) are considered. The P-stability of first-approximation systems is defined. A necessary and sufficient condition for P-stability is that the Lyapunov matrix differential equation should possess a periodic solution. An equivalent form is proposed for this criterion, using which one can reduce the problem of stability for stochastic systems to determining the spectral radius of a certain positive operator. When that is done, lower (upper) bounds for the spectral radius yield necessary (sufficient) conditions for stability. The possibilities of obtaining constructive estimates are demonstrated for a system with one type II noise. A parametric stability criterion, which is a stochastic analogue of the well-known Poincaré criterion, is given for a two-dimensional system (the spectral radius is found in explicit form). Copyright © 1996 Elsevier Science Ltd.

Consider an autonomous system

$$dx = f(x)dt \tag{0.1}$$

where x is an n-vector, $x = \xi(t)$ is a T-periodic solution of system (0.1), which is not the rest point ($f(\xi(t)) \neq 0$), and γ is the phase trajectory (orbit) of this solution. The necessary and sufficient conditions for exponentially orbital stability (the Andronov–Vitt theorem and its analogues [1–3]), expressed in terms of the characteristic indices of the first-approximation system for perturbed motion

$$dy = F(t)ydt, \quad F(t) = \frac{\partial f}{\partial x}(\xi(t)) \tag{0.2}$$

relate to Lyapunov's first method.

The main technique for investigating the stability of stochastic systems in Lyapunov's second method (see [4, 5]). A special construction of Lyapunov functions (LFs) was proposed in [6] to investigate the orbital stability of system (0.1)—orbital Lyapunov functions (OFLs). A brief description of OLFs is given in Section 1. The method of OFLs was used in [7] when analysing the stability of a deterministic orbit γ to random perturbations of the system of Itô stochastic equations

$$dx = f(x)dt + \sum_{r=1}^{m} \sigma_r(x)dw_r(t) \tag{0.3}$$

where $w_r(t)$ ($r = 1, \ldots, m$) are independent standard Wiener processes, and $\sigma_r(x)$ are sufficiently smooth vector-valued functions of the appropriate dimension. To ensure that $x = \xi(t)$ is still a solution of system (0.3), it is assumed that

$$\sigma_r|_\gamma = 0 \tag{0.4}$$

The traditional approach to the choice of LFs in analyses of the stability of the rest point is to take the LF of the corresponding first-approximation system. In [7], however, OFLs were constructed without using first-approximation systems. This paper will introduce constructions of first-approximation systems for the non-linear system (0.3) and investigate their stability.

First-approximation systems are introduced (Section 2) in connection with the approximation of the generating differential operator of system (0.3) on the class of OFLs (Section 1), in two forms (noise of type I or II). Systems with type II noise are easier to set up than systems with type I noise [8]. At the same time, a system with just one type II noise enables one to consider such important cases as the nth-order equation (Section 2), as well as the general two-dimensional system (Section 6).

In addition, a specially selected single type II noise may serve as a majorant for a sequence of m arbitrary type I noises (Section 3).

†*Prikl. Mat. Mekh.* Vol. 60, No. 4, pp. 582–594, 1996.

First-approximation systems constitute a certain class. This class consists of linear stochastic differential equations with periodic coefficients (2.1) and (2.2) that possess a characteristic property: they have a deterministic periodic solution with appropriate degeneracy of the multiplicative noises (2.3) and (2.4). The concept of P-stability will be defined for systems of this class in Section 3. Theorem 1 states that a necessary and sufficient condition for P-stability is that the corresponding Lyapunov matrix differential equation should possess a P-positive-definite T-periodic solution. The solution of this equation may be found by the build-up (see Theorem 2) and iteration methods (Section 4). The results of Sections 2 and 3 give the stability criterion of the traditional form of a stability theorem based on the first approximation [7, Theorem 1].

It is frequently inconvenient to investigate stability issues by direct examination of the solvability of the Lyapunov matrix equation, particularly in near-critical cases. By using the spectral theory of positive operators it has been possible [8, 9] to devise fairly simple and effective stability criteria for systems with constant coefficients. A similar approach will be extended in Section 4 to investigate the stability of a stochastic system to the determination of the spectral radius ρ of a certain positive operator and verification of the inequality $\rho < 1$. On that basis, lower (upper) bounds for ρ yield necessary (sufficient) conditions for stability. The possibilities of obtaining constructive estimates are demonstrated in Section 5 for a system with one type II noise.

In Section 6 the spectral radius is determined explicitly for the case $n = 2$ (a system in a plane). This enabled a parametric stability criterion to be obtained which is a stochastic analogue of the well-known Poincaré criterion.

1. ORBITAL LYAPUNOV FUNCTION

A sufficiently smooth function is an OLF in a neighbourhood U of an orbit γ if

$$v|_{\gamma} = 0, \quad v|_{U/\gamma} > 0 \tag{1.1}$$

Proofs of the theorem of stability in the first approximation for the case of the rest point utilize quadratic forms as LFs. In the context of orbital stability, functions of the form

$$v(x) = \Delta^T(x)\Phi(\gamma(x))\Delta(x) \tag{1.2}$$

play a similar role, where $\gamma(x)$ is the point nearest to x on γ, $\Delta(x) = x - \gamma(x)$ is a vector representing the deviation of x from γ, $\Phi(\cdot)$ is a function defined on γ such that $\Phi(\cdot)$ for each $x \in \gamma$ is a symmetric $n \times n$ matrix and

$$\Phi(x)r(x) = 0 \tag{1.3}$$

where $r(x)$ is the vector tangent γ at x [6]. It is natural to call (1.2) an orbital quadratic form.

The functions $\Phi(\cdot)$ may be associated, using a solution $x = \xi(t)$ defining a natural parametrization of the curve γ, with the elements V of a certain space Σ. The elements of Σ are the T-periodic symmetric $n \times n$ matrices $V(t)$, defined and sufficiently smooth on R^1, such that, for any $t \in R^1$

$$V(t)f(\xi(t)) = 0 \tag{1.4}$$

Each function $\Phi(\cdot)$ defines a T-periodic matrix $V(t) = \Phi(\xi(t))$; conversely, every matrix $V(t) \in \Sigma$, through the function $t = t(x)$ inverse to $x = \xi(t)$, defines a function $\Phi(x) = V(t(x))$ on γ. In either case equalities (1.3) and (1.4) follow easily from one another.

As we shall see, the orbital quadratic form $v(x)$ is uniquely defined in a sufficiently small neighbourhood U by the relationship

$$v(x) = \Delta^T(x)V(t(\gamma(x)))\Delta(x) \tag{1.5}$$

given the solution $\xi(t)$ and the matrix $V \in \Sigma$.

The fact that the function (1.5) is positive-definite in the sense of (1.1) is related to the fact that the matrix P is $V(t)$-positive-definite. Consider the matrix $P_y = I - yy^T/(y^Ty)$. The matrix P_y defines a projection operator whose range is a subspace orthogonal to the vector y. We define a T-periodic matrix $P(t) = P_{y(t)}$.

Definition 1 [6]. A T-periodic symmetric matrix $V(t)$ is said to be $P(t)$-positive-definite at time t if, for any vector z such that $P(t)z \neq 0$, we have $z^TV(t)z > 0$. A matrix $V(t)$ which is $P(t)$-positive-definite for any $t \in R^1$ is said to be P-positive-definite.

Let us consider the cone of matrices in the space Σ defined by $K = \{V \in \Sigma \mid V(t)$ is positive-semidefinite for any $t \in R^1\}$ and the set K_P, where $K_P = \{V \in \Sigma \mid V(t)$ is P-positive-definite$\}$.

2. FIRST-APPROXIMATION SYSTEMS

Consider the stochastic systems

$$dz = F(t)zdt + \sum S_r(t)zdw_r \tag{2.1}$$

$$dz = F(t)zdt + \sum \sqrt{(z, Q_r(t)z)}d\eta_r \tag{2.2}$$

where z is an n-vector, $w_r(t)$ $(r = 1, \ldots, m)$ is an independent sequence of standard Wiener processes, and $\eta_r(t)$ $(r = 1, \ldots, m)$ are n-dimensional Wiener processes with parameters

$$Ed\eta_r(t) = 0, \quad Ed\eta_r(t)d\eta_r^T(t) = G_r(t)dt$$

The parameters in (2.1) and (2.2)—the $n \times n$ matrices $F(t)$, $S_r(t)$, $Q_r(t)$, $G_r(t)$—are T-periodic functions and Q_r and G_r are symmetric and positive-semidefinite. Summation is always performed from $r = 1$ to $r = m$.

It is assumed here that some T-periodic vector-valued function $y(t)$ is a deterministic solution of systems (2.1) and (2.2) and

$$S_r(t)y(t) = 0 \tag{2.3}$$

$$Q_r(t)y(t) = 0 \tag{2.4}$$

We shall refer to the noises in system (2.1) as type I noises and to those in system (2.2) as type II noises [8].

Systems (2.1), (2.3) and (2.2), (2.4) arise as first-approximation systems when the generating differential operator L of a non-linear system (0.3) is approximated.

The operator L is defined [5] by

$$Lv(x) = \left(f(x), \frac{\partial v(x)}{\partial x}\right) + \frac{1}{2}\sum \left(\sigma_r(x), \frac{\partial^2 v(x)}{\partial x^2}\sigma_r(x)\right)$$

The approximation Lv, where $v(x)$ is an OLF, has the following form in the neighbourhood of an orbit γ [7]

$$Lv(x) \doteq \Delta^T(x)W(t(x))\Delta(x) \tag{2.5}$$

with $W(t) = \mathcal{L}_1[V(t)]$, where

$$\mathcal{L}_1[V] = V' + F^T V + VF + \sum S_r^T V S_r \tag{2.6}$$

$$V(t) = \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(\xi(t)), \quad F(t) = \frac{\partial f}{\partial x}(\xi(t)), \quad S_r(t) = \frac{\partial \sigma_r}{\partial x}(\xi(t))$$

The operator $\mathcal{L}_1$, defined in the space Σ, is related to the generating differential operator

$$L_1 v(t,z) = \frac{\partial v(t,z)}{\partial t} + \left(F(t)z, \frac{\partial v(t,z)}{\partial z}\right) + \frac{1}{2}\sum \left(S_r(t)z, \frac{\partial^2 v(t,z)}{\partial z^2}S_r(t)z\right)$$

of the linear system (2.1) by the formula

$$L_1(z^T V(t)z) = z^T(\mathcal{L}_1[V(t)])z = z^T W(t)z \tag{2.7}$$

Formulae (2.5)–(2.7) define the role of system (2.1) as a first-approximation system in solving the problem of whether the non-linear system (0.3) is stable. The matrix $V(t)$ simultaneously defines both a LF $v(t,z) = z^T V(t)z$ of system (2.1) and an OLF (1.5) of system (0.3). In the context of orbital stability, as we shall see, the relationship between the Lyapunov functions of the non-linear system and the corresponding first-approximation system is somewhat more difficult to establish than in the case of the rest point.

We know that the function $y(t) = f(\xi(t))$ is a solution of system (0.2). Differentiating the identity $\sigma_r(\xi(t)) \equiv 0$ (see (0.4)) with respect to t, we obtain an identity

$$\frac{\partial \sigma_r}{\partial x}(\xi(t))f(\xi(t)) \equiv 0$$

which means that the matrices S_r of system (2.1) satisfy (2.3). Thus, the first-approximation system (2.1) has a specific property—it has a deterministic solution $y(t)$.

Suppose that the diffusion coefficients in system (0.3) can be written as

$$\sigma_r(x) = \beta_r(x)\varphi_r(x) \tag{2.8}$$

where $\beta_r(x)$ are scalar functions: $\beta_r|_\gamma = 0$ and $\varphi_r(x)$ are n-dimensional vector-valued functions. The function $\beta_r(x)$ defines the intensity of the rth noise, while $\varphi_r(x)$ "distributes" its action over the equations of the system. In that case the parameters S_r of the first-approximation system (2.1) may be written as

$$S_r(t) = p_r(t)q_r(t), \quad p_r(t) = \varphi_r(\xi(t)), \quad q_r(t) = \frac{\partial \beta_r}{\partial x}(\xi(t))$$

At the same time, system (2.2) may also be taken here as the first-approximation system.

Indeed, if (2.8) holds, the matrix $W(t)$ in (2.5) may be written as $W(t) = \mathcal{L}_2[V(t)]$, where

$$\mathcal{L}_2[V] = V' + F^T V + VF + \sum \operatorname{tr}(VG_r)Q_r \tag{2.9}$$

$$G_r(t) = p_r(t)p_r^T(t), \quad Q_r(t) = q_r(t)q_r^T(t) \tag{2.10}$$

the operator $\mathcal{L}_2$ being related by the formula

$$\mathcal{L}_2(z^T V(t)z) = z^T(\mathcal{L}_2[V(t)])z$$

to the generating differential operator L_2 of system (2.2). The parameters of the noises of system (2.2) are related to the coefficients (2.8) of system (0.3) by formulae (2.10), and we can set $\eta_r(t) = w_r(t)q_r(t)$. Differentiating the identity $\beta_r(\xi(t)) = 0$ with respect to t, we obtain an identity $q_r^T(t)y(t) = 0$, which means that the matrices $Q_r(t)$ of system (2.2) satisfy (2.4). Thus, together with a system of type (2.1), which involves type I noises, one can also take a system of type (2.2), with type II noises, as the first-approximation system.

In many important cases, the form of type II noises is more natural for first-approximation systems (see Remark 2 and the example in Section 6). Type II noises are easier to set up than type I noises. This enables us, for example, using one type II noise as a simultaneous majorant for several type I noises, to obtain simple sufficient conditions for the stability of a system with type I noises (see Remark 6).

Remark 1. Suppose that $G_r = G$ $(r = 1, \ldots, m)$ in system (2.2). Then the operator L_2 has the form

$$\mathcal{L}_2[V] = V' + F^T V + VF + \operatorname{tr}(VG)Q, \quad Q = \sum Q_r$$

and it may be implemented using the following system, which involves only one noise

$$dz = F(t)zdt + \sqrt{(z, Q(t)z)}d\eta(t) \tag{2.11}$$

where $\eta(t)$ is an n-dimensional Wiener process with parameters

$$Ed\eta(t) = 0, \quad Ed\eta(t)d\eta^T(t) = G(t)dt$$

Remark 2. Consider the nth-order equation

$$x^{(n)} = g(x, x', \ldots, x^{(n-1)}) + \sum \beta_r(x, x', \ldots, x^{(n-1)})w_r'(t)$$

with a T-periodic solution $x = \xi(t)$: $\beta_r(\xi(t), \ldots, \xi^{(n-1)}(t)) = 0$. Writing this equation as a system of the form (0.3), we get

$$x_1 = x, \quad x_2 = x', \ldots, x_n = x^{(n-1)}$$
$$f_1 = x_2, \ldots, f_{n-1} = x_n, \quad f_n = g(x_1, \ldots, x_n)$$
$$\sigma_r = \beta_r(x_1, \ldots, x_n)\varphi, \quad \varphi = (0, \ldots, 0, 1)^T$$

A first-approximation system in the class of systems with type I noises will be

$$dz = F(t)zdt + \sum \varphi q_r^T x dw_r$$

where

$$F(t) = \begin{Vmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ \dfrac{\partial g}{\partial x_1} & \cdot & \cdot & \ldots & \dfrac{\partial g}{\partial x_n} \end{Vmatrix}, \quad q_r^T(t) = \left(\frac{\partial \beta_r}{\partial x_1}, \ldots, \frac{\partial \beta_r}{\partial x_n} \right)$$

are evaluated along the solution $\xi(t)$. In the class of systems with type II noises (see Remark 1), a suitable first-approximation system will be the single-noise system (2.11), with $Q = \Sigma q_r q_r^T$, $\eta(t) = w(t)\varphi$ and $w(t)$ a scalar standard Wiener process.

3. P-STABILITY OF LINEAR SYSTEMS

Let us investigate the stability of the trivial solution $z = 0$ for systems of type (2.1), (2.3) and (2.2), (2.4). As these systems have a T-periodic solution $y(t) = f(\xi(t))$, the rest point $z = 0$ cannot be exponentially stable in the traditional sense. We shall consider here a weak analogue of such stability, defined in terms of the projection operator $P(t) = P_{y(t)}$ of Section 1.

Definition 2. The trivial solution $z = 0$ of system (2.1) is said to be exponentially P-stable in the mean square if $\alpha > 0$, $L > 0$ exist, such that

$$E\|P(t)z(t)\|^2 \leq Le^{-\alpha t}E\|P(0)z_0\|^2 \tag{3.1}$$

for any initial data $z(0) = z_0$ of the solution $z(t)$ of system (2.1). In such cases we shall say briefly that system (2.1) is P-stable.

To avoid misunderstandings, we note that a similar term occurs in the literature [5], namely, "exponential p-stability" (small p), in relation to the behaviour of moments of the pth power.

Theorem 1. Let system (2.1) be P-stable. Then
(a) for any matrix $C \in K$, the equation

$$\mathcal{L}_1[V] = V' + F^T V + VF + \sum S_r^T V S_r = -C(t) \tag{3.2}$$

has a unique solution in K—a matrix $V \in K$;
(b) if $C \in K_p$, then $V \in K_p$.
Suppose that for some matrix $C \in K_p$ Eq. (3.2) has a solution $V \in K_p$. Then system (2.1) is P-stable.

Proof. Necessity. Consider the function $v(t, z) = z^T V(t)z$ defined by some symmetric matrix $V(t)$. Let $z(t)$ be a solution of system (2.1). It follows from Itô's formula and from (2.7) that

$$\frac{d}{dt}[Ev(t, z(t))] = EL_1 v(t, z(t)) = E[z^T(t)(\mathcal{L}_1[V(t)])z(t)] \tag{3.3}$$

If $V(t)$ is a solution of Eq. (3.2) and $z(\tau) = z$, where z is an arbitrary deterministic vector, then integration of Eq. (3.3) gives

$$E[z^T(t)V(t)z(t)] - z^T V(\tau)z = -\chi(\tau,t) \tag{3.4}$$

$$\chi(\tau,t) = \int_\tau^t E[z^T(s)C(s)z(s)]ds$$

Let $V(s, t)$, $s \in [\tau, t]$, be a solution of the equation $\mathcal{L}_1[V(s)] = C(s)$ such that $V(t, t) = 0$. Then it follows from (3.4) that

$$\chi(\tau,t) = z^T V(\tau,t)z \tag{3.5}$$

For any matrix $C \in K$, the integral in (3.5) is a monotone increasing function of t and, by (3.1), it converges as $t \to \infty$. This means that $V(\tau, t)$ is a monotone increasing function of t and tends to a limit

$$V(\tau) = \lim_{t\to+\infty} V(\tau,t)$$

By (3.5), the limit function $V(\tau)$ satisfies the equality

$$\chi(\tau,\infty) = z^T V(\tau)z \tag{3.6}$$

The function $V(\tau)$ is a solution of Eq. (3.2). Since C is a positive-semidefinite matrix, the same is true of V. Let $z_1(s)$ be a solution of Eq. (2.1) such that $z_1(\tau + T) = z$. Owing to the T-periodicity of the coefficients of Eq. (2.1) and the matrix C, we have

$$\chi(\tau,\infty) = \int_{\tau+T}^{\infty} E[z_1^T(s)C(s)z_1(s)]ds$$

whence it follows that $V(\tau) = V(\tau + T)$. As we see, the limit function $V(\tau)$ is T-periodic. Substituting $z(s) = f(\xi(s))$, $z = f(\xi(\tau))$ into (3.6) and taking into account the fact that $C(s)f(\xi(s)) = 0$, we immediately infer that $V(\tau)f(\xi(\tau)) = 0$. Thus the matrix $V \in K$ is indeed a solution of Eq. (3.2).

We have to prove that the solution is unique. Let $V_1 \in K$ and $V_2 \in K$ be two solutions of Eq. (3.2). The difference $\Delta = V_1 - V_2$ satisfies the homogeneous equation $\mathcal{L}_1[\Delta] = 0$. It then follows from (3.4) that

$$E[z^T(t)\Delta(t)z(t)] = z^T\Delta(\tau)z \tag{3.7}$$

Since system (2.1) is P-stable and $\Delta(t) = P(t)\Delta(t)P(t)$ is bounded as $t \to \infty$, the left-hand side of (3.7) tends to zero. Passing to the limit in (3.7), we obtain $z^T\Delta(\tau)z = 0$, which implies that $\Delta(\tau) = 0$. This completes the uniqueness proof.

Now let $C \in K_p$. Then for any z such that $P(\tau)z \neq 0$, we have $z^TC(\tau)z > 0$, whence it follows that

$$\chi(\tau, \infty) > 0$$

which means that $z^TV(\tau)z > 0$. We have thus proved that $V \in K_p$, completing the proof of necessity.

Sufficiency. Let $V \in K_p$ and $C \in K_p$ be matrices for which Eq. (3.2) is true. Then Eq. (3.3) implies the following identity for any solution $z(t)$ of system (2.1)

$$\frac{d}{dt}E[z^T(t)V(t)z(t)] = -E[z^T(t)C(t)z(t)] \tag{3.8}$$

Since $V, C \in K_p$, $k_i > 0$ ($i = 1, 2, 3$) exist such that

$$k_1V(t) \leqslant C(t) \tag{3.9}$$

$$k_2P(t) \leqslant V(t) \leqslant k_3P(t) \tag{3.10}$$

It follows from (3.8) and (3.9) that

$$E[z^T(t)V(t)z(t)] \leq e^{-k_1 t}E[z^T(0)V(0)z(0)] \tag{3.11}$$

and from (3.10) and (3.11) that

$$E\|P(t)z(t)\|^2 \leq \frac{k_3}{k_2}e^{-k_1 t}E\|P(0)z(0)\|^2$$

implying that system (2.1) is indeed P-stable.

This result is a periodic and "P-projection" version of Theorem 3.2 in [5, Chap. 6].

The solution $V(t)$ of Eq. (3.2) may be found by the build-up method. Consider a sequence of functions $V_n(t)$ defined in the interval $[0, T]$ as follows: $V_1(t)$ is the solution of Eq. (3.2) such that $V_1(t) = B$, where B is an arbitrary symmetric $n \times n$ matrix such that $Bf(\xi(0)) = 0$. The other functions are found by recursion: V_{n+1} is the solution of Eq. (3.2) such that $V_{n+1}(t) = V_n(0)$.

Theorem 2. Let system (2.1) be P-stable. Then the solution $V(t) \in K$ of Eq. (3.2) with $C \in K$ is the limit of the sequence $V_n(t)$: $V(t) = \lim_{n\to\infty} V_n(t)$.

Proof. Let $V(\tau, t)$ be a solution of Eq. (3.2) such that $V(t, t) = B(t)$, where $B(t) = P(t)BP(t)$. It follows that (3.4) that

$$z^T V(\tau,t)z = \chi(\tau,t) + E[z^T(t)B(t)z(t)] \tag{3.12}$$

The analogous equality for $V(\tau)$—the unique solution of Eq. (3.2) in K—is obtained from (3.12) by replacing $B(t)$ by $V(t)$.

Owing to the P-stability of system (2.1), we infer that $V(\tau, t) - V(\tau) \to 0$ as $t \to \infty$, for any z. Consequently, $V(\tau) = \lim_{n\to\infty} V(\tau, t)$.

The statement of the theorem now follows from the obvious relationships

$$V_n(t) = V(t, nT), \quad B(nT) = B$$

As is obvious from the proof of Theorem 2, the build-up method converges at the rate of a geometric progression with quotient $q = e^{-\alpha T}$, where α is the index of exponential decrease in (3.1).

Remark 3. The results obtained here for systems with type I noise also hold for systems with type II noise. In that case system (2.1), (2.3) and Eq. (3.2) in Definition 2 and Theorems 1 and 2 should be replaced respectively by system (2.2), (2.4) and the equation

$$\mathscr{L}_2[V] = V' + F^T V + VF + \sum \operatorname{tr}(VG_r)Q_r = -C \tag{3.13}$$

4. SPECTRAL CRITERION FOR P-STABILITY

Let us write Eqs (3.2) and (3.13) in a unified notation

$$\mathscr{L}[V] = -C, \quad \mathscr{L} = \mathscr{A} + \mathscr{S} \tag{4.1}$$

where $\mathscr{A}$ is a differential operator, related to the deterministic part of system (2.1), (2.2) by the formula

$$\mathscr{A}[V] = V' + F^T V + VF \tag{4.2}$$

and $\mathscr{S}$ is an operator related to the corresponding stochastic parts of the systems. We then have, for (2.1)

$$\mathscr{S}[V] = \mathscr{S}_1[V] = \sum S_r^T V S_r \tag{4.3}$$

and for (2.2)

$$\mathscr{S}[V] = \mathscr{S}_2[V] = \sum \operatorname{tr}(VG_r)Q_r \tag{4.4}$$

In this notation

$$\mathcal{L}_l = \mathcal{A} + \mathcal{S}_l, \quad l = 1, 2$$

A necessary condition for systems (2.1) and (2.2) to be P-stable is that the deterministic system (0.2) must be P-stable. On the assumption that system (0.2) is P-stable, it follows from Theorem 1(a) and from the fact that K is a reproducing cone in the space Σ that the operator $\mathcal{A}$, considered on Σ, has an inverse $\mathcal{A}^{-1}$, and moreover that $\mathcal{A}^{-1}$ is negative. Note that in both cases (4.3) and (4.4) the operator $\mathcal{S}$ is positive.

Applying the operator $\mathcal{A}^{-1}$ to both sides of Eq. (4.1), we obtain

$$V - \mathcal{P}[V] = -\mathcal{A}^{-1}[C], \quad \mathcal{P} = -\mathcal{A}^{-1}\mathcal{S} \tag{4.5}$$

Thus, we have defined positive operators $\mathcal{P}_1 = -\mathcal{A}^{-1}\mathcal{S}_l$ for systems (2.1) and (2.2). There are analogous constructions for stochastic systems with constant coefficients [8, 9]. The use of the spectral theory of positive operators has enabled us to obtain fairly constructive stability criteria. In this paper we extend that approach to systems with periodic coefficients.

Theorem 3. System (2.1) ((2.2)) is P-stable if and only if

$$\rho(\mathcal{P}) < 1 \tag{4.6}$$

where $\rho(\mathcal{P})$ is the special radius of the operator $\mathcal{P}$.

Proof. Necessity. Let system (2.1) ((2.2)) be P-stable. Then system (0.2) is also P-stable, guaranteeing the existence of the operator $\mathcal{A}^{-1}$. Proceeding as before (see Theorem 1), we deduce from Eq. (4.1) for some $V \in K_p$, $C \in K_p$ that Eq. (4.5) holds, from which it follows, in view of $-\mathcal{A}^{-1}[C] \in K_P$, that $V - \mathcal{P}[V] \in K_P$. The operator $\mathcal{P}$, as the product of the two positive operators $-\mathcal{A}^{-1}$ and $\mathcal{S}$, is also positive. Now, using Theorem 16.7 of [10], we immediately arrive at (4.6).

Sufficiency. As already pointed out, the P-stability of system (0.2) guarantees the existence of $-\mathcal{A}^{-1}$ and, together with it, of $\mathcal{P}$. Due to condition (4.6), the operator $\mathcal{B}$ defined by $\mathcal{B}[V] = V - \mathcal{P}[V]$ has an inverse and moreover $\mathcal{B}^{-1} = \Sigma_{k=0}^{\infty}\mathcal{P}^k$, i.e. $\mathcal{B}^{-1}$ is positive. This means that for $C \in K_p$ the matrix $V = \mathcal{B}^{-1}[-\mathcal{A}^{-1}[C]] \in K_P$ is a solution of Eq. (4.5). Hence, by the equivalence of (4.5) and (4.1), it follows that $V \in K_p$ satisfies Eq. (4.1). Consequently (see Theorem 1), system (2.1) ((2.2)) is P-stable.

Remark 4. In the system

$$dz = F(t)z\,dt + \varepsilon \sum S_r(t)z\,dw_r \tag{4.7}$$

where the constant $\varepsilon > 0$ defines the intensity of the interference, the quantity $\rho(\mathcal{P})$ determines the critical value $\varepsilon^* = \sqrt{(1/\rho(\mathcal{P}))}$ of the parameter ε at which system (4.7) ceases to be P-stable. When $\rho(\mathcal{P}) = 0$, system (4.7) is P-stable for any ε.

Remark 5. It follows from the proof of Theorem 3 that the matrix $V(t)$—the solution of Eq. (4.1)—is the limit of the monotone increasing sequence of matrices $V_n(t)$: $V_0(t) = -\mathcal{A}^{-1}[C]$, $V_n = \Sigma_{k=0}^{n}\mathcal{P}^k[V_0]$. These matrices V_n may be found iteratively: $V_{n+1} = \mathcal{P}[V_n] + V_0$. In circumstances such that the solution of the deterministic Lyapunov equation (the determination of the values of the operator $\mathcal{A}^{-1}$) is a fairly easy procedure and the stochastic system has a sufficient reserve of stability (the spectral radius of $\mathcal{P}$ is far from unity), the iterative method provides an effective algorithm for solving Eq. (4.1).

Theorem 3 reduces the problem of the stability of a stochastic system to determining the spectral radius ρ of the operator $\mathcal{P}$ and checking for the condition $\rho < 1$. In this situation, lower (upper) bounds for the spectral radius yield necessary (sufficient) conditions for stability.

5. BOUNDS OF THE SPECTRAL RADIUS OF THE OPERATOR $\mathcal{P}$ FOR A SYSTEM WITH A SINGLE TYPE II NOISE

Consider the system

$$dz = F(t)zdt + \sqrt{(z, Q(t)z)}d\eta \tag{5.1}$$

where $\eta(t)$ is an n-dimensional Wiener process with parameters $Ed\eta(t) = 0$, $Ed\eta(t)d\eta^T(t) = G(t)dt$, $G \in K$, $Q \in K_P$. It is assumed that the deterministic part (system (0.2)) is P-stable, i.e. that $\mathcal{A}^{-1}$ exists. The operator $\mathcal{S}$ for (5.1) is

$$\mathcal{S}[V] = \mathrm{tr}(VG)Q \tag{5.2}$$

The positive operator $\mathcal{P} = -\mathcal{A}^{-1}\mathcal{S}$ has a spectral radius ρ that is an eigenvalue with eigenvector $V \in K$ (see Theorem 11.5 in [10]). In view of (5.2), we can write the relationship $\mathcal{P}[V] = \rho V$ as

$$-\mathcal{A}^{-1}[\mu(t)Q(t)] = \rho V(t), \quad \mu(t) = \mathrm{tr}(V(t)G(t)) \geqslant 0 \tag{5.3}$$

where $\mu(t)$ is a T-periodic function. It follows from (5.3) that

$$\mathcal{B}[\mu] = \rho\mu \tag{5.4}$$

where $\mathcal{B}[\delta] = -\mathrm{tr}(\mathcal{A}^{-1}[\delta(t)Q(t)]G(t))$ is a positive operator on the cone of non-negative T-periodic scalar functions $\delta(t)$.

Here $\mu(t)$ is an eigenfunction of the operator $\mathcal{B}$, and $\rho(\mathcal{B}) = \rho(\mathcal{P}) = \rho$. The simple structure of the operator $\mathcal{S}$ in the case of a single type II noise (see (5.2)) has made it possible to change from $\mathcal{P}$ to $\mathcal{B}$, at the same time lowering the dimension of the problem to be solved.

Let us assume (normalization condition) that $\int\mu(t)dt = 1$ (throughout, unless otherwise specified, the integration will be performed from $t = 0$ to $t = T$). It then follows from (5.4) that

$$\rho = \int\mathcal{B}[\mu]dt = -\langle\mathcal{A}^{-1}[\mu Q], PGP\rangle \tag{5.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product defined in Σ by

$$\langle V, W\rangle = \int\mathrm{tr}(VW)dt$$

Passing to adjoints in (5.5), we obtain

$$\rho = \int\mu\,\mathrm{tr}(QD)dt = -\langle\mu Q, (\mathcal{A}^*)^{-1}[PGP]\rangle \tag{5.6}$$

where $D(t)$ is a T-periodic solution of the equation

$$\mathcal{A}^*[D] = -D' + FD + DF^T = -PGP \tag{5.7}$$

From (5.6) we derive simple bounds for ρ

$$\min_{[0,T]}\mathrm{tr}(QD) \leqslant \rho \leqslant \max_{[0,T]}\mathrm{tr}(QD) \tag{5.8}$$

Note that (5.7) is the equation for the second moment $Ez(t)z^T(t)$ of the system

$$dz = F(t)zdt + P(t)d\eta \tag{5.9}$$

obtained from (5.1) by replacing the multiplicative noise by an appropriate additive noise. In this situation, irrespective of the choice of initial data, the matrix of second moments of system (5.9) converges to the T-periodic matrix $D(t)$, which determines both the spectral radius (5.6) itself and the bounds (5.8).

We now consider another approach to estimating the spectral radius. Let $W(t) = -\mathcal{A}^{-1}[Q] \in K_P$ be a solution of the equation

$$W' + F^TW + WF = -Q \tag{5.10}$$

Let $q_1(t) > 0, q_2(t) > 0$ be T-periodic functions related to the matrices $Q(t)$ and $W(t)$ by the inequalities

$$q_1(t)W(t) \leqslant Q(t) \leqslant q_2(t)W(t) \tag{5.11}$$

Such functions always exist. They have been used [11] to obtain bounds for the characteristic exponents.

Consider the functions

$$\begin{aligned} &\alpha(\rho,t) = \mathrm{tr}(W(t)G(t)) - \rho \\ &\varphi_1(\rho,t) = q_1(t)\alpha^+(\rho,t) + q_2(t)\alpha^-(\rho,t) \\ &\varphi_2(\rho,t) = q_2(t)\alpha^+(\rho,t) + q_1(t)\alpha^-(\rho,t) \\ &\alpha^\pm = (\alpha \pm |\alpha|)/2, \quad I_l(\rho) = \int \varphi_l(\rho,t)dt, \quad l = 1,2 \end{aligned} \tag{5.12}$$

The functions $I_l(\rho)$ are continuous and have different signs at the endpoints of the interval $[m, M]$, where $m = \min_{[0,T]}\mathrm{tr}(W(t)G(t))$, $M = \max_{[0,T]}\mathrm{tr}(W(t)G(t))$. Let ρ_l be a root of the function $I_l(\rho)$.

Theorem 4. Assume that the deterministic system (0.2) is P-stable and the inequalities (5.11) hold. Then the spectral radius $\rho(\mathcal{P})$ satisfies the inequalities

$$\rho_1 \leqslant \rho(\mathcal{P}) \leqslant \rho_2 \tag{5.13}$$

If the numbers q_1 and q_2 of (5.11) are constants, we have the following bounds

$$\frac{q_1}{q_2}J \leqslant \rho(\mathcal{P}) \leqslant \frac{q_2}{q_1}J, \quad J = \frac{1}{T}\int \mathrm{tr}(WG)dt \tag{5.14}$$

Proof. The functions

$$\mu_l(t) = \exp\left(-\frac{1}{\rho_l}\int_0^t \varphi_l(\rho_l,t)dt\right) > 0$$

are T-periodic solutions of the equations

$$\rho_l\mu_l' + \mu_l\varphi_l(\rho_l,t) = 0 \tag{5.15}$$

It follows from inequality (5.11) for the functions $\varphi_l(\rho, t)$ that

$$\varphi_1(\rho,t)W(t) \leqslant \alpha(\rho,t)Q(t) \leqslant \varphi_2(\rho,t)W(t) \tag{5.16}$$

Relations (5.15) and (5.16) imply inequalities which, in view of (5.12), are equivalent to

$$\rho_1\mathcal{A}[V_1] + \mathcal{S}[V_1] \geqslant 0, \quad \rho_2\mathcal{A}[V_2] + \mathcal{S}[V_2] \leqslant 0 \tag{5.17}$$

where $V_l(t) = \mu_l(t)W(t)$. Inequalities (5.17), in turn, are equivalent to

$$\mathcal{P}[V_1] \geqslant \rho_1 V_1, \quad \mathcal{P}[V_2] \leqslant \rho_2 V_2$$

and these imply inequality (5.13) (see Theorems 16.1, 16.2 in [10]).

Consider the case in which $q_1(t)$ and $q_2(t)$ in inequality (5.11) are constants, i.e.

$$q_1 W(t) \leqslant Q(t) \leqslant q_2 W(t) \tag{5.18}$$

Express the functions φ_l in (5.12) in the form

$$\varphi_1(\rho,t) = q_1\alpha(\rho,t) + (q_2 - q_1)\alpha^-(\rho,t), \quad \varphi_2(\rho,t) = q_2\alpha(\rho,t) + (q_1 - q_2)\alpha^-(\rho,t)$$

The inequalities

$$(q_2 - q_1)\alpha^-(\rho,t) \geq -(q_2 - q_1)\rho, \quad (q_1 - q_2)\alpha^-(\rho,t) \leq -(q_1 - q_2)\rho$$

imply the inequalities

$$\varphi_1(\rho,t) \geq q_1 \operatorname{tr}(WG) - q_2\rho = \varphi_1^*, \quad \varphi_2(\rho,t) \leq q_2 \operatorname{tr}(WG) - q_1\rho = \varphi_2^*$$

which in turn imply the inequalities (see (5.12))

$$I_1^*(\rho) \leq I_1(\rho), \quad I_2^*(\rho) \geq I_2(\rho), \quad I_l^*(\rho) = \int \varphi_l^*(\rho,t)dt \tag{5.19}$$

In view of (5.19), the roots ρ_l^* of the functions $I_l^*(\rho)$

$$\rho_1^* = \frac{q_1}{q_2}J, \quad \rho_2^* = \frac{q_2}{q_1}J$$

are related to the roots ρ_l of the functions $I_l(\rho)$ by the inequalities

$$\rho_1^* \leq \rho_1, \quad \rho_2 \leq \rho_2^*$$

Thus, using (5.13), we obtain (5.14).

Remark 6. System (5.1) with one type II noise may be used as a majorant for system (2.1), which involves several type II noises. Indeed, the inequality $S_r^T V S_r \leq \operatorname{tr}(VS_rS_r^T)P_1$, which holds for any matrix $V \in K$, implies that

$$\mathcal{G}_1[V] = \sum S_r^T V S_r \leq \operatorname{tr}(VG)P = \mathcal{G}_2[V], \quad G = \sum S_r S_r^T$$

The operators $\mathcal{P}_l = -\mathcal{A}^{-1}\mathcal{G}_l$ satisfy the inequality $\mathcal{P}_1 \leq \mathcal{P}_2$, from which it follows that $\rho(\mathcal{P}_1) \leq \rho(\mathcal{P}_2)$. Thus, the P-stability of system (5.1) with $Q = P$, $\eta = \Sigma S_r w_r$ is a sufficient condition for the P-stability of system (2.1).

6. EXAMPLE

Consider system (2.1) in the case when $n = 2$. The projection matrix will then be of rank one and may be written as $P(t) = v(t)v^T(t)$, where $v(t)$ is a normalized vector, orthogonal for any t to the vector $y(t) = f(\xi(t))$. It follows from conditions (2.3) that the matrices S_r may be represented as $S_r = b_r v^T(t)$, where $b_r = S_r v$. In view of this structure of S_r, the m type I noises of system (2.1) may be replaced by a single type II noise. As a result, system (2.1) is replaced by the equivalent system

$$dz = F(t)zdt + \sqrt{z^T P z}\, d\eta, \quad \eta(t) = \sum b_r w_r(t) \tag{6.1}$$

The matrix V, playing the role of an eigenvector of the operator $\mathcal{P}$ of system (6.1), is also of rank one and it may be written as $V(t) = \mu(t)P(t)$, where $\mu(t)$ is a T-periodic scalar function. The relationship $\mathcal{P}[V] = \rho V$ (where ρ is the spectral radius of $\mathcal{P}$) leads to the following equation for $\mu(t)$

$$\rho[\mu' P + \mu P' + \mu(F^T P + PF)] + \operatorname{tr}(GP)P = 0 \tag{6.2}$$

$$G = \sum b_r b_r^T = \sum S_r S_r^T$$

Multiplying Eq. (6.2) on the left by v^T and on the right by v, using the equalities $v^T P v = (v^T v)^2 = 1$, $v^T P' v = (v^T v)' = 0$, we obtain the equation

$$\rho(\mu' + \alpha(t)\mu) + \beta(t)\mu = 0 \tag{6.3}$$

where

$$\alpha(t) = v^T(F^T + F)v, \quad \beta(t) = v^T G v \tag{6.4}$$

Dividing (6.3) by $\mu \neq 0$ and integrating over $[0, T]$, we get

$$\rho = -\int \beta(t)dt\left[\int \alpha(t)dt\right]^{-1}$$

—the unique eigenvalue of $\mathcal{P}$. The inequality

$$\int \alpha(t)dt < 0 \tag{6.5}$$

is a necessary and sufficient condition for the deterministic part of system (6.1) to be P-stable. In view of the equality

$$\int \alpha(t)dt = 2\int \operatorname{tr} F dt$$

condition (6.5) is equivalent to the well-known inequality (the Poincaré criterion, see [2])

$$\lambda = T^{-1}\int \operatorname{tr} F dt < 0$$

where λ is a characteristic exponent of the system $dz = F(t)zdt$. Note that, since S_r is degenerate

$$\beta(t) = \operatorname{tr}\left(\sum S_r(t)S_r^T(t)\right)$$

Thus, the inequality $\rho < 1$ (the necessary and sufficient condition for the P-stability of system (2.1)) may be written as follows:

$$\int \operatorname{tr}(2F(t) + \sum S_r(t)S_r^T(t))dt < 0$$

REFERENCES

1. ANDRONOV A. A. and VITT A. A., On Lyapunov stability. *Zh. Eksp. Teor. Fiz.* **2,** 5, 373–374, 1933.
2. DEMIDOVICH B. P., *Lectures on the Mathematical Theory of Stability.* Nauka, Moscow, 1967.
3. HARTMAN P. H., *Ordinary Differential Equations.* John Wiley, New York, 1964.
4. KATS I. Ya. and KRASOVSKII N. N., The stability of systems with random parameters. *Prikl. Mat. Mekh.* **24,** 5, 809–823, 1960.
5. KHAS'MINSKII R. Z., *The Stability of Systems of Differential Equations with Randomly Perturbed Parameters.* Nauka, Moscow, 1969.
6. MIL' SHTEIN G. N., The stability of stabilization of periodic motions of autonomous systems. *Prikl. Mat. Mekh.* **41,** 4, 744–749, 1977.
7. MIL' SHTEIN G. N. and RYASHKO L. B., The stability and stabilization of orbits of autonomous systems under random perturbations. *Prikl. Mat. Mekh.* **56,** 6, 951–958, 1992.
8. RYASHKO L. B., Stabilization of linear stochastic systems with perturbations depending on the state and the control. *Prikl. Mat. Mekh.* **43,** 4, 612–620, 1979.
9. LEVIT M. V. and YAKUBOVICH V. A., An algebraic criterion for stochastic stability of linear systems with parametric inference of the white-noise type. *Prikl. Mat. Mekh.* **36,** 1, 142–148, 1972.
10. KRASNOSEL'SKII M. A., LIFSHITS Ye. A. and SOBOLEV A. V., *Positive Linear Systems.* Nauka, Moscow, 1985.
11. YAKUBOVICH V. A. and STARZHINSKII V. M., *Linear Differential Equations with Periodic Coefficients and Their Applications.* Nauka, Moscow, 1972.

Translated by D.L.